Random Processes and Growth of Firms

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Lack of relation between theoretical and empirical investigations. . .

Economics consists of theoretical laws which nobody has verified and of empirical laws which nobody can explain.
Economic distributions as steady-state equilibrium:

- certain economic distributions are stable over time
- we are aware of a continuing movement of the elements which make up the population in question

This suggests the idea of steady-state equilibrium: “a state of macroscopic equilibrium maintained by a large number of transitions in opposite directions” (Feller, 1957)
Let us consider two “economic” populations

<table>
<thead>
<tr>
<th>HUMAN BEINGS</th>
<th>FIRMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>total population</td>
<td>total number of firms</td>
</tr>
<tr>
<td>age-structure</td>
<td>size distribution</td>
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- total population
- age-structure
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**FIRMS**
- total number of firms
- size distribution
- expected gains/losses or ruin plus entry
1. Introduction

2. The Pioneers:
   - Jacobus Cornelius Kapteyn and Robert Gibrat
   - Michal Kalecki
   - Herbert Simon

3. The Law finding Process

4. An empirically based model of firm growth
The Pioneers
Empirical starting point: firm size distribution conforms approximately to normality once plotted on a log scale.

To explain how this distribution arises Kapteyn and Gibrat proposed the famous Law of Proportionate Effect:

\[ X_t - X_{t-1} = \epsilon_t X_{t-1} \quad \epsilon \sim i.i.d.(\mu, \sigma^2) \]  

Two possible interpretations:

1. in a process of growth equal proportionate increments have the same chance of occurring in a given time-interval whatever size happens to have reached.

2. growth in proportion to size is a random variable with a given distribution assumed constant over time.
This simple model corresponds to a special kind of stochastic process called Random Walk, in our case it is a RW in logs.

In the most elementary case of RW we consider discrete equal time-interval and an object which wanders on an infinite straight line taking at each time a unitary step leftward or rightward with probability p and 1-p respectively.

The RW considered by Kapteyn and Gibrat is slightly more complicated: the size of the step taken is itself a random variable.

**Economic interpretation**: essentially no structure in the process of firm growth.
At this point two main weaknesses of the “unrestricted” Gibrat’s model deserve to be highlighted

- theoretically, the variance of size $t\sigma^2$ explodes for $t \to \infty$

- empirically, we do not observe any increase in the dispersion of the size distribution

To cope with this drawback it is necessary to introduce a stability condition to offset the tendency to diffusion. The type of condition chosen and its interpretation become the distinctive feature of various theories emanating from the Gibrat’s model.
Kalecki started from the observation that the variance of the size of all business firms remains constant over time

\[
\frac{1}{N} \sum (X_t + g_t)^2 = \frac{1}{N} \sum (X_t)^2 ,
\]

\[\downarrow\]

\[2 \sum X_t g_t = - \sum g_t^2 \]

i.e. the proportionate random increment is negatively correlated with size. He assumed a linear relation between \( g_t \) and \( X_t \)

\[
g_t = -\alpha X_t + z_t
\]

proving that the distribution of \( X_t \) is a Normal.
The Islands Model

- The market consists of a number of independent submarkets (islands)
- Each market is large enough to support exactly one plant
- There exists a set of preexisting business opportunities or equivalently there’s a constant arrival of new opportunities
- These opportunities are independent each other
- Firm’s size is measured by the number of opportunities it has taken up
Herbert Simon is the father of the so-called “Empirically Based Industrial Dynamics”.

- RW based on the Gibrat’s Law generates the Log-Normal distribution only if all the elements in the population starts the “walk” at the same time. Is that plausible? Simon consider a different stochastic process in which new entrants are an integral part of the process itself. A possible critique.

- Economic theory has little to say about the distribution of firm sizes:
  1. static cost theory (constant or U-shaped cost curves) provides no predictions
  2. Bain(1956) suggests that above some critical minimum cost curve for the firm shows virtually constant return to scale
Simon’s model: assumptions

1. There is a minimum size, $s_m$, of firms in an industry.

2. Size has no effect upon the expected percentage growth of a firm:
   - empirically observed
   - implied by another empirical fact: constant returns to scale above a certain minimal threshold (Bain).

3. New firms are being born in the smallest size-class at a constant rate.

Under these assumptions the steady-state distribution of the process is:

$$ f(s) = \rho B(s, \rho + 1) \quad \text{Yule distribution} \quad (5) $$

and

$$ \lim_{s \to \infty} f(s) = \rho \Gamma(\rho + 1) s^{-(\rho+1)} \quad \text{Pareto tail} \quad (6) $$

Remark: the entry process is crucial!
The Yule distribution (but also the Log-Normal) generally fits the data quite well.

The observed frequencies are pareto in the upper tail: Yule distribution is OK not the Log-Normal.

Gibrat’s Law seems verified by the data. The story is not simple:

1. Weak and strong Gibrat’s law
2. Sectoral disaggregation
Concentration ratio: when one fits a distribution function to observed data on the basis of a theoretical model it is reasonable to ground his measure of concentration on the parameters of the distribution function.

In the Simon model there is only one parameter \( \rho \)

\[
\rho = \frac{1}{1 - \frac{G_N}{G}} \quad (7)
\]

\( G_N \) is the share of growth of new firms.

Reinterpreting the Gibrat’s process. This implies that the same equilibrium distribution can be obtained with various degrees of mixing, i.e. with various amounts of firm mobility among size classes.
Critique on the robustness of any empirical regularity on size distribution: “All families of distributions tried so far fail to describe at least some industry well” (Schmalansee, 1989).

He considers the theoretical framework developed by Simon reversing the question: can we put any restrictions on the shape of the size distribution?

Rejection of the Gibrat’s Law in favour of a weaker hypothesis: the probability that the next market opportunity is taken by any currently active firm is non-decreasing in the size of that firm.

Under these assumption a lower bound to concentration is derived and used to empirically validate the model.
The Law Finding Process
The Law Finding Process (i.e. “Retroduction”)

1. Looking for facts

2. Finding simple generalizations that describe the facts to some degree of approximation

3. Finding Limiting conditions under which the deviations of facts from generalization might be expected to decrease

4. Explaining why the generalization “should” fit the facts

5. The explanatory theories generally make predictions that go beyond the simple generalizations and hence suggest new empirical tests.
Empirically based Industrial Dynamics

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An Empirically Based Model of Firm Growth
The Databases

**COMPUSTAT** U.S. publicly traded firms in the Manufacturing Industry (SIC code ranges between 2000-3999) in the time window 1982-2001. We have 1025 firms in 15 different two digit sectors.

**MICRO.1** Developed by the Italian Statistical Office (ISTAT). More than 8000 firms with 20 or more employees in 97 sectors (3-digit ATECO) in the time window 1989-1996. We use 55 sectors with > 44 firms.
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Firms Size

We consider \( S_{ij}(t) \) is the size of firm \( i \) in sector \( j \) at time \( t \). We define the normalized (log) size

\[
 s_{ij}(t) = \log(S_{ij}(t)) - <\log(S_{ij}(t))>_i \quad (8)
\]

Main results on empirical firms size densities

1. Heterogeneity of shapes across sectors
2. Bimodality and no log-normality
3. Separation core-fringe
4. Paretian upper-tails?
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Empirical Size Densities - US

Aggregate

![Graph of Empirical Size Densities - US](image)
Empirical Size Densities - US

Aggregate

Food

Apparel
Empirical Size Densities - ITA

Aggregate

Pharmaceuticals

Cutlery, tools and general hardware
Empirical Size Densities - ITA

Aggregate

Pharmaceuticals

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Footwear
Firms Growth Rates

We build firms growth rates as the first difference of $S_{ij}$

$$g_{ij}(t) = s_{ij}(t) - s_{ij}(t - 1)$$  \hspace{0.5cm} (9)

Main results on empirical growth rates densities

1. shape is stable over time
2. display similar shapes across sectors
3. look similar to the Laplace
4. present similar width(?)
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Empirical Growth Rates Densities - US

Aggregate

Food
Empirical Growth Rates Densities - US
Empirical Growth Rates Densities - ITA

 Aggregate

![Graph showing empirical growth rates densities for ITA. The x-axis ranges from -1.5 to 1.5, and the y-axis ranges from 0.01 to 1.0. The graph displays the distribution of growth rates with a peak at 0 and tails extending to -1.5 and 1.5 on the x-axis.]
The Subbotin Distribution

\[ f_S(x) = \frac{1}{2ab^{1/b}\Gamma(1/b + 1)} e^{-\frac{1}{b} \left| \frac{x-\mu}{a} \right|^b} \]  

(10)

Mikhail Fyodorovich Subbotin (1883-1966)
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ML Estimation Procedure

We consider:

\[- \log(L_S(x; a, b, \mu)) = \]

\[
n \log \left(2b^{1/b} a \Gamma(1 + 1/b) \right) + (ba^b)^{-1} \sum_{i=1}^{n} |x_i - \mu|^b \quad (11)\]

and we minimize it with respect to the parameters using a multi-step procedure.

These ML estimators are asymptotically consistent in all the parameter space, asymptotically normal for \( b > 1 \) and asymptotically efficient for \( b > 2 \).
## Estimates on Italian Sectors

<table>
<thead>
<tr>
<th>Ateco code</th>
<th>Sector</th>
<th>Parameter $b$</th>
<th>Parameter $a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>151</td>
<td>Production, processing and preserving of meat</td>
<td>0.83</td>
<td>0.05</td>
</tr>
<tr>
<td>155</td>
<td>Dairy products</td>
<td>0.91</td>
<td>0.07</td>
</tr>
<tr>
<td>158</td>
<td>Production of other foodstuffs (bread, sugar, etc...)</td>
<td>0.89</td>
<td>0.05</td>
</tr>
<tr>
<td>159</td>
<td>Production of beverages (alcoholic and not)</td>
<td>0.88</td>
<td>0.06</td>
</tr>
<tr>
<td>171</td>
<td>Preparation and spinning of textiles</td>
<td>1.19</td>
<td>0.07</td>
</tr>
<tr>
<td>172</td>
<td>Textiles weaving</td>
<td>1.12</td>
<td>0.06</td>
</tr>
<tr>
<td>173</td>
<td>Finishing of textiles</td>
<td>1.11</td>
<td>0.06</td>
</tr>
<tr>
<td>175</td>
<td>Carpets, rugs and other textiles</td>
<td>1.02</td>
<td>0.08</td>
</tr>
<tr>
<td>177</td>
<td>Knitted and crocheted articles</td>
<td>0.97</td>
<td>0.05</td>
</tr>
<tr>
<td>182</td>
<td>Wearing apparel</td>
<td>0.92</td>
<td>0.03</td>
</tr>
<tr>
<td>191</td>
<td>Tanning and dressing of leather</td>
<td>1.12</td>
<td>0.09</td>
</tr>
<tr>
<td>193</td>
<td>Footwear</td>
<td>1.12</td>
<td>0.05</td>
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<tr>
<td>202</td>
<td>Production of plywood and panels</td>
<td>0.98</td>
<td>0.09</td>
</tr>
<tr>
<td>203</td>
<td>Wood products for construction</td>
<td>0.94</td>
<td>0.08</td>
</tr>
<tr>
<td>205</td>
<td>Production of other wood products (cork, straw, etc...)</td>
<td>1.31</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Estimates on US Sectors

<table>
<thead>
<tr>
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<th>Parameter $b$</th>
<th>Parameter $a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>Food and kindred products</td>
<td>0.9888</td>
<td>0.7039</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0010</td>
<td>0.0005</td>
</tr>
<tr>
<td>23</td>
<td>Apparel and other textile products</td>
<td>1.0819</td>
<td>0.7664</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0027</td>
<td>0.0013</td>
</tr>
<tr>
<td>26</td>
<td>Paper and allied products</td>
<td>1.0999</td>
<td>0.7663</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0024</td>
<td>0.0011</td>
</tr>
<tr>
<td>27</td>
<td>Printing and publishing</td>
<td>0.9621</td>
<td>0.7115</td>
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<td></td>
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<td>0.0015</td>
<td>0.0008</td>
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<tr>
<td>28</td>
<td>Chemicals and allied products</td>
<td>1.0164</td>
<td>0.7562</td>
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<td></td>
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<td>0.0004</td>
<td>0.0002</td>
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<tr>
<td>29</td>
<td>Petroleum and coal products</td>
<td>1.1841</td>
<td>0.8370</td>
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<td></td>
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<td>0.0043</td>
<td>0.0019</td>
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<tr>
<td>30</td>
<td>Rubber and miscellaneous plastics products</td>
<td>0.9487</td>
<td>0.7148</td>
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<tr>
<td></td>
<td></td>
<td>0.0018</td>
<td>0.0010</td>
</tr>
<tr>
<td>32</td>
<td>Stone, clay, glass, and concrete products</td>
<td>1.1023</td>
<td>0.7720</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0039</td>
<td>0.0018</td>
</tr>
<tr>
<td>33</td>
<td>Primary metal industries</td>
<td>1.1254</td>
<td>0.7870</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0015</td>
<td>0.0007</td>
</tr>
<tr>
<td>34</td>
<td>Fabricated metal products</td>
<td>0.9081</td>
<td>0.6639</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0013</td>
<td>0.0007</td>
</tr>
<tr>
<td>35</td>
<td>Industrial machinery and equipment</td>
<td>0.9466</td>
<td>0.6761</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0003</td>
<td>0.0002</td>
</tr>
<tr>
<td>36</td>
<td>Electrical and electronic equipment</td>
<td>0.8989</td>
<td>0.6303</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>37</td>
<td>Transportation equipment</td>
<td>1.0033</td>
<td>0.7107</td>
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<tr>
<td></td>
<td></td>
<td>0.0011</td>
<td>0.0005</td>
</tr>
<tr>
<td>38</td>
<td>Instruments and related products</td>
<td>0.9722</td>
<td>0.6980</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0004</td>
<td>0.0002</td>
</tr>
<tr>
<td>39</td>
<td>Miscellaneous manufacturing industries</td>
<td>1.0232</td>
<td>0.7447</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0022</td>
<td>0.0011</td>
</tr>
</tbody>
</table>
The Theoretical Framework

Observed growth as the cumulative effect of diverse “events”

\[ g(t; T) = s(t + T) - s(t) = \epsilon_1(t) + \epsilon_2(t) + \ldots = \sum_{j=1}^{G(t; T)} \epsilon_j(t) \]  

- The Gibrat Tradition: \( \epsilon_j \) are r.v. independent from size \( s \) (strong form: \( \epsilon_j \) are i.i.d.) Limitation: No interaction among firms
- The “Islands” Models: Simon introduces Finite number of \( M \) opportunities progressively captured by \( N \) firms. \( G(t; T) \) becomes a r.v. Limitation: Equipartition of opportunities among firms → Gaussian growth rates
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The Model

Multi-step simulation model

Business Events → Micro-Shocks → Growth

Self-reinforcing effect in events assignment. Idea of “competition among objects whose market success...[is] cumulative or self-reinforcing” (B.W. Arthur)

Discrete time stochastic growth process; at each round a two steps procedure is implemented:

- determine the number of events captured by a firm, $G(t; T)$
- disclose $\epsilon_j$, $j = \{1, \ldots, G(t; T)\}$, i.e. the effect of these events on firm size
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- disclose $\epsilon_j, \ j = \{1, \ldots, G(t; T)\}$, i.e. the effect of these events on firm size
1. Consider an urn with $N$ different balls, each representing a firm.

   Draw a ball and replace with **TWO** of the same kind. (Here the first draw from an urn with two types of ball)

2. Repeat this procedure $M$ times

RESULT: partition of $M$ events on $N$ firms.
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Draw a ball and replace with TWO of the same kind. (Here the first draw from an urn with two types of ball)

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STEP 2 - The Generation of Shocks

From the previous assignment procedure

\[ m_i(t) = \text{# of opportunity given to firm i at time t} \]

A very simple relation between “opportunities” and growth:

\[ s_i(t + T) - s_i(t) = \sum_{j=1}^{m_i(t)+1} \epsilon_j(t) \]  \hspace{1cm} (13)

\( \epsilon \) are i.i.d. with a common distribution \( f(\epsilon) \).

Run the simulation and collect statistics.
Simulation Results

Growth rates densities for $N = 100$ and different values of $M$.
We define $D = |F_{\text{model}}(x; M, N) - F_L(x)|$ the absolute deviation between the empirical growth rates distribution (as approximated by the Laplace) and the distribution predicted by the model. Here $D$ as a function of the number of firms $N$ and the average number of micro-shocks per firm $M/N$. 
Why does the Model work?

The unconditional growth rates distribution implied by this model is given by

\[ \sum_{h=0}^{M} P(h; N, M) F(x; v_0)^{(h+1)} \]

Events Distribution  Distribution of the sum of \( h \) micro-shocks

(14)

In the assignment procedure above \( P \) follows a Bose-Einstein

\[ P(h; N, M) = \frac{P(X)}{P(X|m_1 = h)} = \frac{\binom{N+M-h-2}{N-2}}{\binom{N+M-1}{N-1}} \]

while follows a Binomial in the Simon tradition.
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Bose-Einstein and binomial with $N = 100$ and $M = 10,000$. 
Theorem

Suppose that the micro-shocks distribution possesses the second central moment $\sigma^2_\epsilon < \infty$. Under the Polya opportunities assignment procedure the firms growth rates distribution converges in the limit for $N, M \to \infty$ to a Laplace distribution with parameter $\sqrt{\nu}/2$, i.e.

$$\lim_{M, N \to \infty} f_{\text{model}} = f_L(x; \sqrt{\nu}/2) = \frac{1}{\sqrt{2\nu}} e^{-\sqrt{2/\nu} |x|}$$

where $\nu = \sigma^2_\epsilon M/N$. 
A new stylized fact has been presented

We show its robustness under disaggregation

Our original explanation is based on a general mechanism of short-horizon "dynamic increasing returns" in a competitive environment

We provide a "Large Industry" Limit Theorem

Simulations show that "Large" is not so large
For my work see www.sssup.it/secchi and www.cafed.eu

Here some selected ref:

Consider STATES succeeding on another in time. The state is, for example, the yearly sales of a firm.

The state changes at discrete intervals of time only and remains fixed. Each state is fully determined by the preceding state and by a random element.

The functioning of any particular Markov Chain is represented by a matrix of probabilities of transitions.

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Gibrat’s Law: the probability of jumping from one sales class to another depends only on the width of the jump.